Lecture 3:

Last time: Two analytical methods

1. Integrating factor method for first order differential eqt

2. Integrating factor method for second order differential eqt

Example: Solve:
$$\begin{cases} \dfrac{dy}{dx} + \dfrac{2}{x} y = x - 1 + \dfrac{1}{x} & \text{for } 1 < x < \infty \\ y(1) = \dfrac{1}{2} & \text{(Boundary condition)} \end{cases}$$

Solution: Step 1: Compute the integrating factor

$$M(x) = e^{\int P(x)\,dx} = e^{\int \frac{2}{x}} = e^{2\ln x} = x^2$$

Step 2: Multiply both side by M(x)

$$M(x)\left(\frac{dy}{dx} + \frac{2}{x} y\right) = M(x)\left(x - 1 + \frac{1}{x}\right)$$

$$\Leftrightarrow \frac{d}{dx}\Big(M(x)\,y(x)\Big) = x^2\left(x - 1 + \frac{1}{x}\right)$$

$$\Leftrightarrow \frac{d}{dx}\Big(x^2 y(x)\Big) = x^3 - x^2 + x$$

Step 3: Integrate both sides

$$x^2 y(x) = \int x^3 - x^2 + x\,dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + C \Leftrightarrow y(x) = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{C}{x^2}$$

Step 4: Substitute boundary condition

$$y(1) = \frac{1}{2} \Leftrightarrow y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = \frac{1}{2} \Leftrightarrow C = \frac{1}{12} \quad \therefore y(x) = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{1}{12x^2} \;//$$

Example: Solve:
$$\begin{cases} -2\dfrac{d^2u}{dx^2} + 4u(x) = 4x^2 - 4x + 12 \quad \text{for } 0 < x < 1 \quad (*) \\ u(0) = 1 \quad \text{and} \quad u(1) = 1 \end{cases}$$

Step 1: Solve the homogeneous eqt first: $-2\dfrac{d^2u}{dx^2} + 4u(x) = 0$

- Multiply both sides by $M(x) = \dfrac{du}{dx}$

$$-2\frac{du}{dx}\frac{d^2u}{dx^2} + 4u(x)\frac{du}{dx} = 0 \iff \frac{d}{dx}\left(-2\left(\frac{du}{dx}\right)^2\right) + \frac{d}{dx}\left(4(u(x))^2\right) = 0$$

- Guess possible solutions:

$$-2\left(\frac{du}{dx}\right)^2 + 4(u(x))^2 = 0 \iff 2\left(\frac{du}{dx}\right)^2 = 4u(x)^2 \iff \frac{du}{dx} = \pm\sqrt{2}\,u(x) \text{ are possible solutions}$$

$\iff u_1(x) = \alpha_1 e^{\sqrt{2}x}$ and $u_2(x) = \alpha_2 e^{-\sqrt{2}x}$ are possible solutions

$\therefore u(x) = \alpha_1 e^{\sqrt{2}x} + \alpha_2 e^{-\sqrt{2}x}$ (for some α_1 and α_2) is a solution for $(*)$

Step 2: Guess a particular solution $w(x)$

Guess: $w(x) = a_2 x^2 + a_1 x + a_0$ and put it into $(*)$.

We get: $-2(2a_2) + 4(a_2x^2 + a_1x + a_0) = 4x^2 - 4x + 12 \iff 4a_2x^2 + 4a_1x + 4a_0 - 4a_2 = 4x^2 - 4x + 12$

$\iff 4a_2 = 4\ ;\ 4a_1 = -4\ ;\ 4a_0 - 4a_2 = 12 \iff a_2 = 1,\ a_1 = -1,\ a_0 = 4$

$\therefore w(x) = x^2 - x + 4$ is a particular solution.

Step 3: Construct general sols and substitute boundary conditions

General sols: $u(x) = \alpha_1 e^{\sqrt{2}x} + \alpha_2 e^{-\sqrt{2}x} + (x^2 - x + 4)$ is a general sol because:

$$\underbrace{\left[-2\frac{d^2}{dx^2}\left(\alpha_1 e^{\sqrt{2}x}+\alpha_2 e^{-\sqrt{2}x}\right)+4\left(\alpha_1 e^{\sqrt{2}x}+e^{-\sqrt{2}x}\right)\right]}_{=\,0}+\underbrace{\left[-2\frac{d^2}{dx^2}\left(x^2-x+4\right)+4\left(x^2-x+4\right)\right]}_{4x^2-4x+12}=4x^2-4x+12$$

Boundary conditions: $\left(u(x)=\alpha_1 e^{\sqrt{2}x}+\alpha_2 e^{-\sqrt{2}x}+4x^3-4x+12\right)$

$$u(0)=\alpha_1+\alpha_2+12=1$$

$$u(1)=\alpha_1 e^{\sqrt{2}}+\alpha_2 e^{-\sqrt{2}}+12=1$$

$$\Longleftrightarrow\begin{cases}\alpha_1+\alpha_2=-11\\ \alpha_1 e^{\sqrt{2}}+\alpha_2 e^{-\sqrt{2}}=-11\end{cases}\quad \text{(Linear system)}$$

$\Downarrow$

Determine α_1 and α_2 (Exercise)

Another useful technique: Separation of variables

Consider a heat equation (on a unit circle):

$$u_t = u_{xx}, \quad x \in [0, 2\pi], \quad t \geq 0$$

subject to: $\begin{cases} u(0,t) = u(2\pi, t) & \text{(periodic condition)} \\ u(x,0) = \sin x & \text{(initial condition)} \end{cases}$

Strategy: Let $u(x,t) = X(x)\,T(t)$.

$$u_t = u_{xx} \Rightarrow X(x)\,T'(t) = X''(x)\,T(t)$$

$$\therefore \quad \frac{X''}{X} = \frac{T'}{T} = \lambda \leftarrow \text{some constant}$$

In particular, $T' = \lambda T$ for some constant λ.

$$\Rightarrow \frac{d}{dt}(\ln T) = \lambda \Rightarrow \ln T = \lambda t + C_0 \Rightarrow T = Ce^{\lambda t}$$

for some constant C and λ.

For X, since $u(x,0) = \sin x$. We may guess $X(x) = \sin x$

Note that $X'' = (-1)\sin x = (-1)X$. $\therefore \lambda = -1$.

Hence a possible solution is of the form:

$$u(x,t) = Ce^{-t}\sin x$$

$$\because u(x,0) = \sin x = C\sin x \Rightarrow C = 1.$$

$$\therefore u(x,t) = e^{-t}\sin x \text{ is a solution.}$$

(multi-variable)

Remark: Separate $u(x,t) = X(x)\,T(t) \rightsquigarrow$ PDE converted to 2 ODEs (single variable)

single variable function

Spectral method

We'll discuss: (1) Analytic (Fourier) Spectral method

(2) Numerical Spectral method

Consider: Analytic (Fourier) Spectral method first !!

Consider general differential eqt:

$$L\,u(x) = g(x) \quad \text{for some differential operator } L$$

$$\left(\text{e.g. } L = \frac{d^2}{dx^2} \quad \text{or} \quad L = \frac{d^2}{dx^2} + \frac{d}{dx} \quad \text{etc} \ldots \right)$$

Example: Consider $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x + 2\cos x$ where $y(0) = y(2\pi)$ (periodic). Find a possible solution of the 2nd order ODE.

Note that: $L = \frac{d^2}{dx^2} + \frac{d}{dx}$ in our case.

Solution:

Consider: $\phi_n(x) = \sin nx$ and $\varphi_n(x) = \cos nx$

Note that: $L\phi_n(x) = \sum_{k=0}^{N} \alpha_k \phi_k(x) + \sum_{k=0}^{N} \beta_k \varphi_k(x)$

(Linear combination of $\phi_k(x)$'s and $\varphi_k(x)$'s

$$L(\sin nx) = \frac{d^2 \sin nx}{dx} + \frac{d}{dx}\sin nx$$

$$= -n^2 \sin nx + n\cos nx$$

(linear combination of $\{\phi_n(x)\}_{n=1}^{\infty}$ and $\{\varphi_n(x)\}_{n=0}^{\infty}$

Let $y(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$

Then: $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin x + 2\cos x$ implies:

$$\sum_{n=1}^{N} \left(-a_n n^2 \cos nx - b_n n^2 \sin nx\right) + \sum_{n=1}^{N} \left(-n a_n \sin nx + n b_n \cos nx\right) = \sin x + 2\cos x$$

$$\Rightarrow \sum_{n=1}^{N} \left[(n b_n - n^2 a_n)\cos nx - (n a_n + n^2 b_n)\sin nx\right] = \sin x + 2\cos x$$

Comparing coefficient: $b_1 - a_1 = 2$, $a_1 + b_1 = -1$ $\Rightarrow$ $b_1 = 1/2$, $a_1 = -3/2$ (Algebraic eqt)

$a_k = b_k = 0$ otherwise

$\therefore$ A possible solution is: $y(x) = -\frac{3}{2}\cos x + \frac{1}{2}\sin x$

Spectral method

Main idea: Consider : $L u(x) = g(x)$ such that u and g are periodic functions (i.e. $u(x+2\pi) = u(x)$, $g(x+2\pi) = g(x)$)

where L is a linear differential operator (e.g. $L = \frac{d^2}{dx^2}$; $L = \frac{d^2}{dx^2} + \frac{d}{dx}$ etc)

(e.g. if $L = \frac{d^2}{dx^2} + \frac{d}{dx}$, then $L u(x) = \frac{d^2u}{dx^2}(x) + \frac{du}{dx}(x)$)

L is linear means : $L\big(u(x) + \underset{\in \mathbb{R}}{a}\, v(x)\big) = L u(x) + a L v(x)$

Assume that $\{\phi_n(x)\}_{n=1}^{\infty}$ are functions such that:

(1) $\phi_n(x)$ is periodic;

(2) $L\phi_n(x)$ is a linear combination $\{\phi_n(x)\}_{n=1}^{\infty}$

Assume: $u(x) \approx \sum_{k=0}^{N} a_k \phi_k(x)$ and $g(x) \approx \sum_{k=0}^{N} b_k \phi_k(x)$

(Note: in solving the differential equation, a_k's are unknown, b_k's are known)

Then: $\phi_n(x)$ is called the basis functions for the differential equation $Lu(x) = g(x)$.

For the ease of explanation, suppose $L\phi_n(x) = \lambda_n \phi_n(x)$.

($\phi_n(x)$ is an eigenfunction of L)

<u>Goal</u>: Find a_k's solving $Lu(x) = g(x)$.

Then: $Lu(x) = g(x)$ implies: $L\left(\sum_{k=0}^{N} a_k \phi_k(x)\right) = \sum_{k=0}^{N} b_k \phi_k(x)$

$$\Rightarrow \sum_{k=0}^{N} a_k L\phi_k(x) = \sum_{k=0}^{N} b_k \phi_k(x)$$

$$\Rightarrow \sum_{k=0}^{N} a_k \lambda_k \phi_k(x) = \sum_{k=0}^{N} b_k \phi_k(x).$$

Comparing coefficients: $a_k \lambda_k = b_k$ (algebraic equation)

$$\therefore a_k = \frac{b_k}{\lambda_k}$$

Thus, the solution is: $u(x) = \sum_{k=0}^{N} \left(\frac{b_k}{\lambda_k}\right) \phi_k(x)$.